Algebra II

18 July 2012

Instructions. All Questions carry equal marks. In the following, F denotes a field.

1. Let V and W be vector spaces over F having dimension n and m respectively. Prove that $n < m$ if and only if there exists a surjective linear transformation from W to V.

2. Find the dimension of the vector space consisting of all 4×4 symmetric matrices with entries from F.

3. Let p be a prime number and let V be a 3 dimensional vector space over the finite field $\mathbb{Z}/p\mathbb{Z}$. Prove that the additive group $(V, +)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

4. Let V be a vector space over F. Prove that the set of all linear transformations from V to F under pointwise addition and scalar multiplication is a vector space isomorphic to V.

5. Let V,W be two finite dimensional vector spaces over a field F. Let $T : V \to W$ be a linear transformation. Prove that

$$\dim(Ker(T)) + \dim(Im(T)) = \dim(V).$$

6. Let T be linear transformation on a finite dimensional vector space over the field of complex numbers $\mathbb{C}$. If 0 is the only eigen value of T, then prove that $T^n = 0$ for some n.

7. Let $\mathbb{R}$ denote the field of real numbers. Find an orthonormal basis of $\mathbb{R}^2$ with respect to the bilinear form X^tAY, where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

8. Let V be a vector space over the field of real numbers $\mathbb{R}$ with a positive definite, symmetric biliear form. Prove that for any subspace W of V, we have $(W^{\perp})^{\perp} = W$.

9. Prove that eigenvectors associated to distinct eigenvalues of a Hermitian matrix A are orthogonal.

10. Prove that a real symmetric matrix A is positive definite if and only if is eigenvalues are positive.